2.3 Quicksort

- quicksort
- selection
- duplicate keys
- system sorts

Two classic sorting algorithms

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

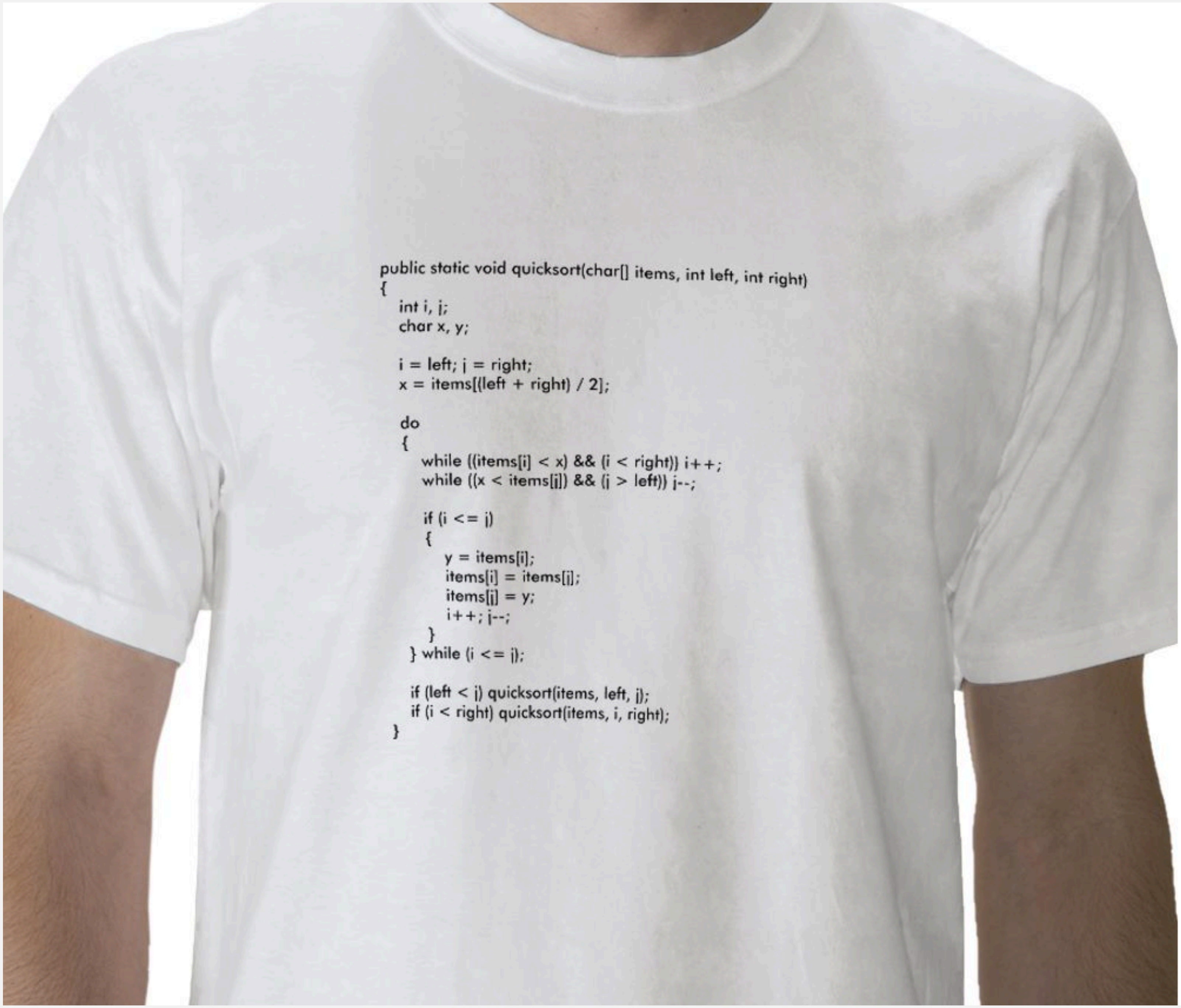
Mergesort. ← last lecture

- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

Quicksort. ← this lecture

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

Quicksort t-shirt

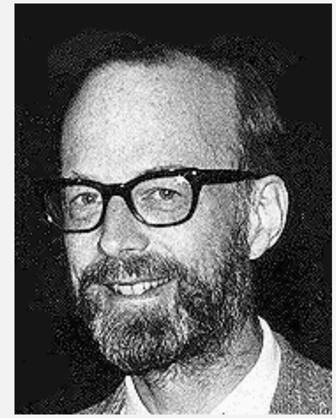


- quicksort
- selection
- duplicate keys
- system sorts

Quicksort

Basic plan.

- Shuffle the array.
- Partition so that, for some `j`
  - entry `a[j]` is in place
  - no larger entry to the left of `j`
  - no smaller entry to the right of `j`
- Sort each piece recursively.



Sir Charles Antony Richard Hoare
1980 Turing Award

input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

partitioning item

not greater *not less*

Quicksort partitioning demo

Quicksort partitioning

Basic plan.

- Scan `i` from left for an item that belongs on the right.
- Scan `j` from right for an item that belongs on the left.
- Exchange `a[i]` and `a[j]`.
- Repeat until pointers cross.

			v								a[i]							
	i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values	0	16	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
scan left, scan right	1	12	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
exchange	1	12	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
scan left, scan right	3	9	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
exchange	3	9	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
scan left, scan right	5	6	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
exchange	5	6	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
scan left, scan right	6	5	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
final exchange	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
result	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Partitioning trace (array contents before and after each exchange)

Quicksort: Java code for partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{
   int i = lo, j = hi+1;
   while (true)
   {
      while (less(a[++i], a[lo]))
         if (i == hi) break;

      while (less(a[lo], a[--j]))
         if (j == lo) break;

      if (i >= j) break;
      exch(a, i, j);
   }

   exch(a, lo, j);
   return j;
}
```

find item on left to swap

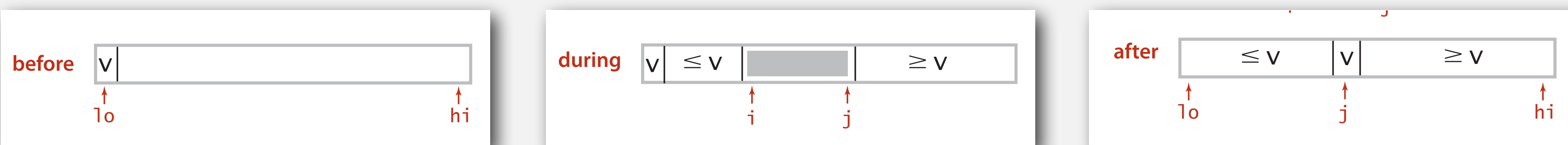
find item on right to swap

check if pointers cross

swap

swap with partitioning item

return index of item now known to be in place



Quicksort: Java implementation

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {  /* see previous slide */  }

   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
   }
}
```

shuffle needed for performance guarantee (stay tuned)

Quicksort trace

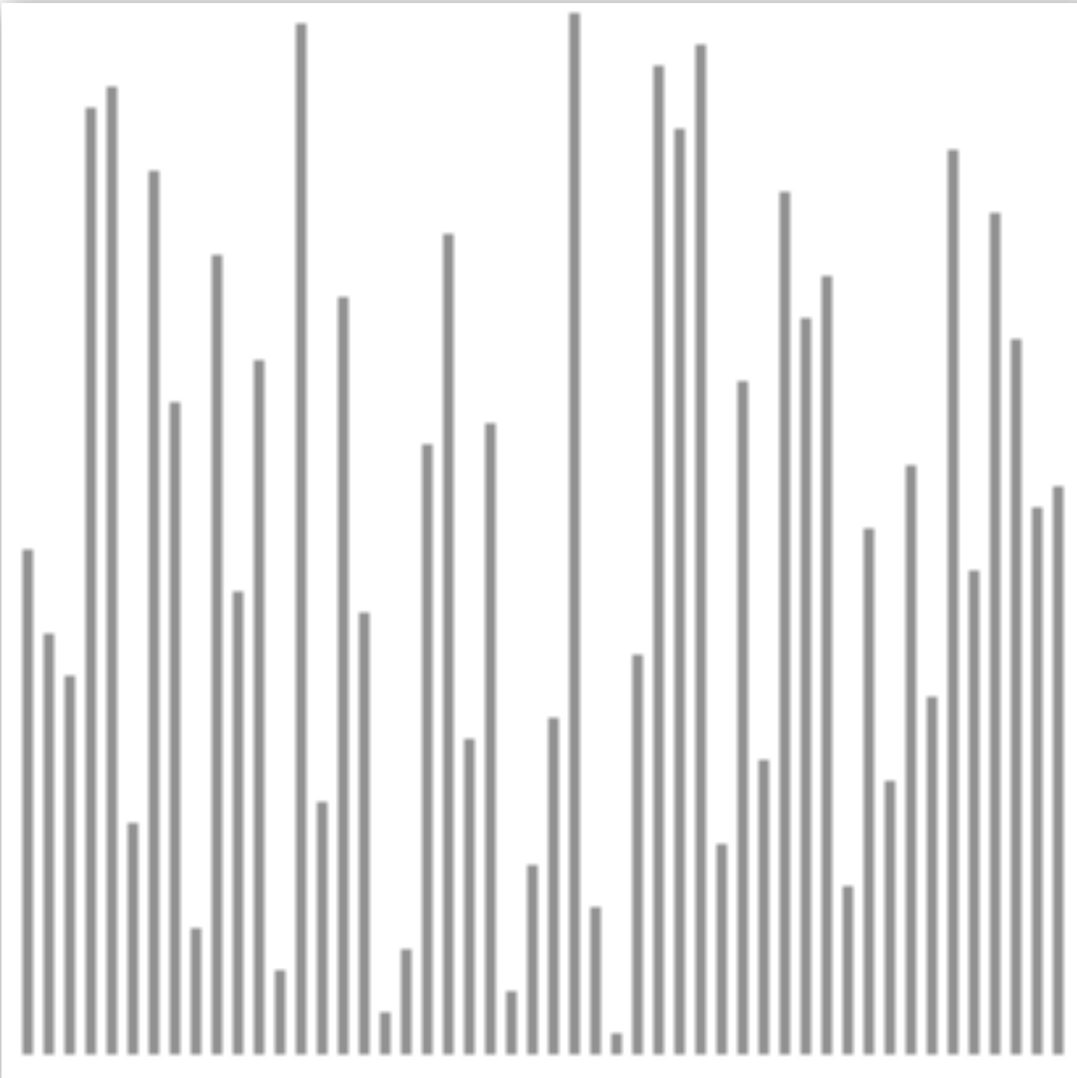
	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort trace (array contents after each partition)

Quicksort animation

50 random items



http://www.sorting-algorithms.com/quick-sort

algorithm position

in order

current subarray

not in order

Quicksort: implementation details

Partitioning in-place. Using an extra array makes partitioning easier (and stable), but is not worth the cost.

Terminating the loop. Testing whether the pointers cross is a bit trickier than it might seem.

Staying in bounds. The `(j == lo)` test is redundant (why?), but the `(i == hi)` test is not.

Preserving randomness. Shuffling is needed for performance guarantee.

Equal keys. When duplicates are present, it is (counter-intuitively) better to stop on keys equal to the partitioning item's key.

Quicksort: empirical analysis

Running time estimates:

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)			quicksort (N log N)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

Quicksort: best-case analysis

Best case. Number of compares is $\sim N \lg N$.

lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: worst-case analysis

Worst case. Number of compares is $\sim \frac{1}{2} N^2$.

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 1. C_N satisfies the recurrence $C_0 = C_1 = 0$ and for $N \geq 2$:

$$C_N = \overset{\text{partitioning}}{(N+1)} + \left(\frac{C_0 + C_{N-1}}{N}\right) + \left(\frac{\overset{\text{left}}{C_1} + \overset{\text{right}}{C_{N-2}}}{\underset{\text{partitioning probability}}{N}}\right) + \ldots + \left(\frac{C_{N-1} + C_0}{N}\right)$$

- Multiply both sides by N and collect terms:

$$NC_N = N(N+1) + 2(C_0 + C_1 + \ldots + C_{N-1})$$

- Subtract this from the same equation for $N - 1$:

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

- Rearrange terms and divide by $N(N+1)$:

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

Quicksort: average-case analysis

- Repeatedly apply above equation:

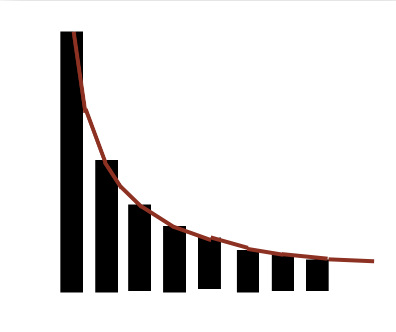
$$\begin{aligned}
\frac{C_N}{N+1} &= \frac{C_{N-1}}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{C_{N-3}}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1} \\
&= \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots + \frac{2}{N+1}
\end{aligned}$$

previous equation

substitute previous equation

- Approximate sum by an integral:

$$\begin{aligned}
C_N &= 2(N+1)\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{N+1}\right) \\
&\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx
\end{aligned}$$



- Finally, the desired result:

$$C_N \sim 2(N+1)\ln N \approx 1.39 N \lg N$$

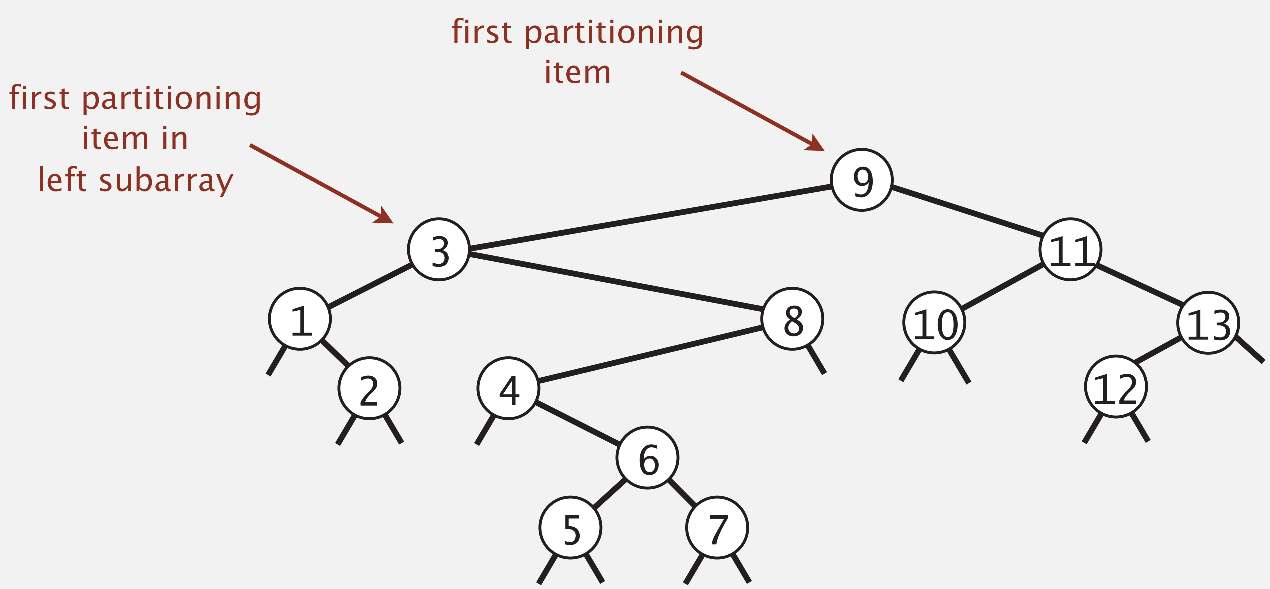
Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 2. Consider BST representation of keys 1 to N.

shuffle

9	10	2	5	8	7	6	1	11	12	13	3	4



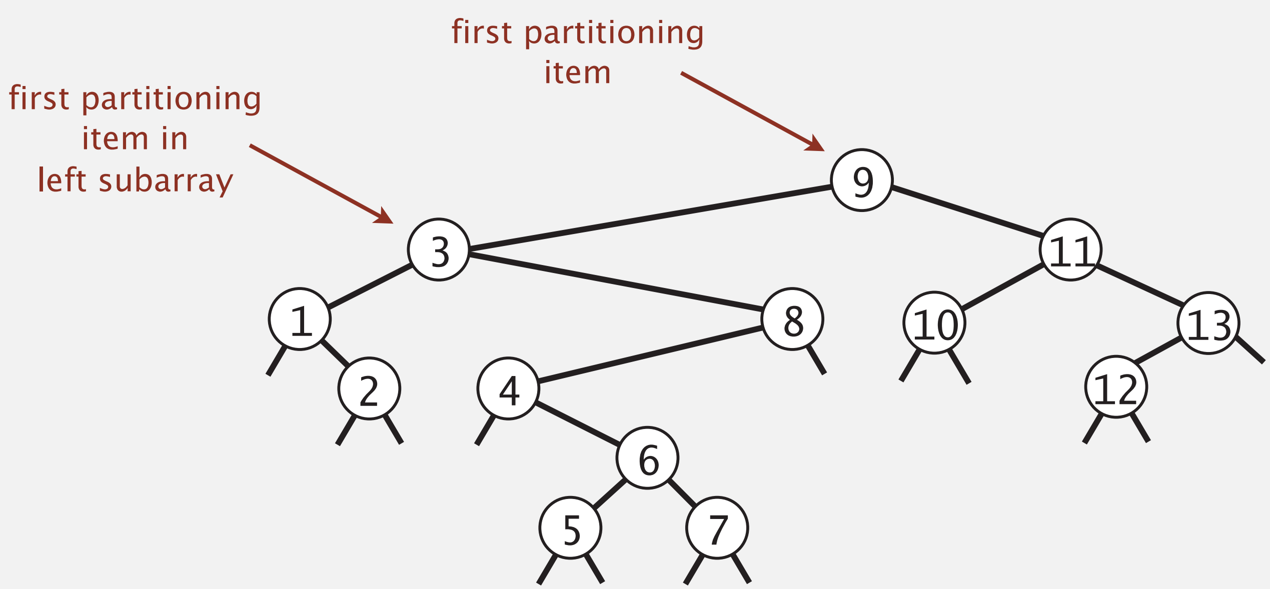
Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 2. Consider BST representation of keys 1 to N.

- A key is compared only with its ancestors and descendants.
- Probability i and j are compared equals $2 / |j - i + 1|$.

3 and 6 are compared (when 3 is partition)

1 and 6 are not compared (because 3 is partition)



Quicksort: average-case analysis

Proposition. The average number of compares C_N to quicksort an array of N distinct keys is $\sim 2N \ln N$ (and the number of exchanges is $\sim \frac{1}{3} N \ln N$).

Pf 2. Consider BST representation of keys 1 to N.

- A key is compared only with its ancestors and descendants.
- Probability i and j are compared equals $2 \,/\, |j - i + 1|$.
- Expected number of compares = $\displaystyle\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{2}{j-i+1}$

all pairs i and j

$$
\begin{aligned}
\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{2}{j-i+1} &= 2 \sum_{i=1}^{N} \sum_{j=2}^{N-i+1} \frac{1}{j} \\
&\leq 2N \sum_{j=1}^{N} \frac{1}{j} \\
&\sim 2N \int_{x=1}^{N} \frac{1}{x}\, dx \\
&= 2N \ln N
\end{aligned}
$$

Quicksort: summary of performance characteristics

Worst case. Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \ldots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39\, N \lg N$.

- 39% more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Caveat emptor. Many textbook implementations go quadratic if array

- Is sorted or reverse sorted.
- Has many duplicates (even if randomized!)

Quicksort properties

Proposition. Quicksort is an in-place sorting algorithm.

Pf.

- Partitioning: constant extra space.
- Depth of recursion: logarithmic extra space (with high probability).

can guarantee logarithmic depth by recurring on smaller subarray before larger subarray

Proposition. Quicksort is not stable.

Pf.

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

Quicksort: practical improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort: practical improvements

Median of sample.

- Best choice of pivot item = median.
- Estimate true median by taking median of sample.
- Median-of-3 (random) items.

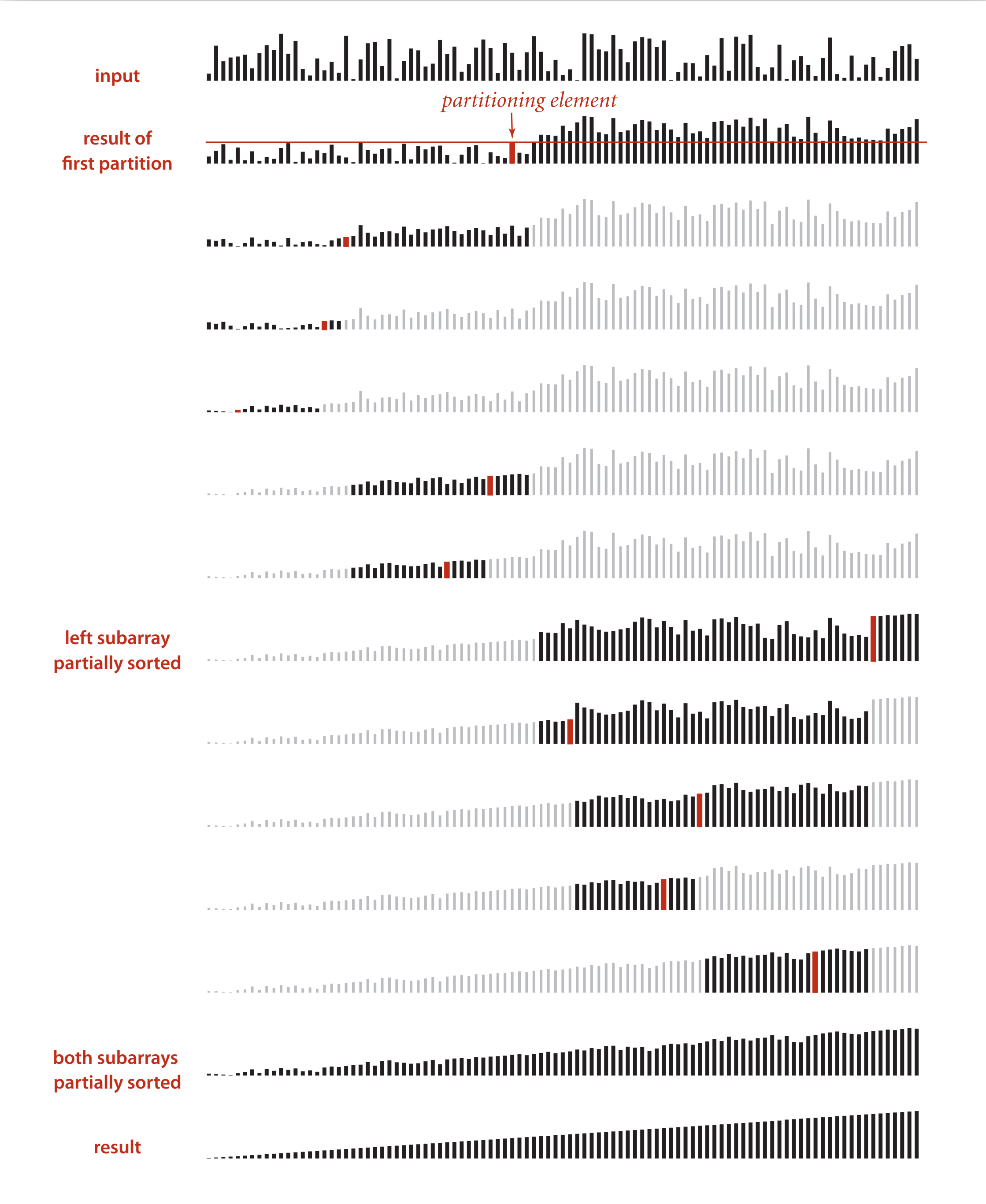
~ 12/7 N ln N compares (slightly fewer)
~ 12/35 N ln N exchanges (slightly more)

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;

   int m = medianOf3(a, lo, lo + (hi - lo)/2, hi);
   swap(a, lo, m);

   int j = partition(a, lo, hi);
   sort(a, lo, j-1);
   sort(a, j+1, hi);
}
```

Quicksort with median-of-3 and cutoff to insertion sort: visualization



- quicksort
- **selection**
- duplicate keys
- system sorts

Selection

Goal. Given an array of N items, find the k^{th} largest.

Ex. Min $(k = 0)$, max $(k = N - 1)$, median $(k = N/2)$.

Applications.

- Order statistics.
- Find the "top k."

Use theory as a guide.

- Easy $N \log N$ upper bound. How?
- Easy N upper bound for $k = 1, 2, 3$. How?
- Easy N lower bound. Why?

Which is true?

- $N \log N$ lower bound? ← is selection as hard as sorting?
- N upper bound? ← is there a linear-time algorithm for each k?

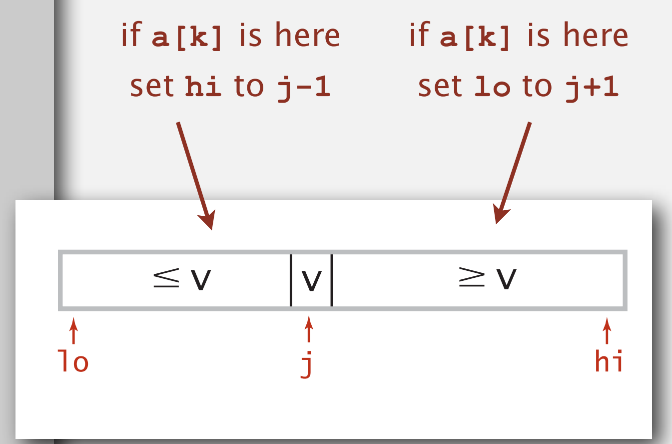
Quick-select

Partition array so that:

- Entry `a[j]` is in place.
- No larger entry to the left of `j`.
- No smaller entry to the right of `j`.

Repeat in one subarray, depending on `j`; finished when `j` equals `k`.

```
public static Comparable select(Comparable[] a, int k)
{
    StdRandom.shuffle(a);
    int lo = 0, hi = a.length - 1;
    while (hi > lo)
    {
       int j = partition(a, lo, hi);
       if      (j < k) lo = j + 1;
       else if (j > k) hi = j - 1;
       else            return a[k];
    }
    return a[k];
}
```



Quick-select: mathematical analysis

Proposition. Quick-select takes linear time on average.

Pf sketch.

- Intuitively, each partitioning step splits array approximately in half: $N + N/2 + N/4 + \ldots + 1 \sim 2N$ compares.
- Formal analysis similar to quicksort analysis yields:

$$C_N = 2\,N + k \ln(N/k) + (N-k) \ln(N/(N-k))$$

(2 + 2 ln 2) N to find the median

Remark. Quick-select uses $\sim \frac{1}{2} N^2$ compares in the worst case, but (as with quicksort) the random shuffle provides a probabilistic guarantee.

Theoretical context for selection

Proposition. [Blum, Floyd, Pratt, Rivest, Tarjan, 1973] There exists a compare-based selection algorithm whose worst-case running time is linear.

Time Bounds for Selection

by

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Remark. But, constants are too high $\Rightarrow$ not used in practice.

Use theory as a guide.

- Still worthwhile to seek practical linear-time (worst-case) algorithm.
- Until one is discovered, use quick-select if you don't need a full sort.

Generic methods

In our `select()` implementation, client needs a cast.

```
Double[] a = new Double[N];
for (int i = 0; i < N; i++)
   a[i] = StdRandom.uniform();
Double median = (Double) Quick.select(a, N/2);
```

unsafe cast
required in client

The compiler complains.

```
% javac Quick.java
Note: Quick.java uses unchecked or unsafe operations.
Note: Recompile with -Xlint:unchecked for details.
```

Q. How to fix?

Generic methods

Pedantic (safe) version. Compiles cleanly, no cast needed in client.

generic type variable
(value inferred from argument `a[]`)

```
public class QuickPedantic
{
    public  static <Key extends Comparable<Key>> Key select(Key[] a, int k)
    {  /* as before */  }
```

return type matches array type

```
    public  static <Key extends Comparable<Key>> void sort(Key[] a)
    {  /* as before */  }

    private static <Key extends Comparable<Key>> int partition(Key[] a, int lo, int hi)
    {  /* as before */  }

    private static <Key extends Comparable<Key>> boolean less(Key v, Key w)
    {  /* as before */  }

    private static <Key extends Comparable<Key>> void exch(Key[] a, int i, int j)
    {  Key swap = a[i]; a[i] = a[j]; a[j] = swap;  }

}
```

can declare variables of generic type

http://www.cs.princeton.edu/algs4/23quicksort/QuickPedantic.java.html

Remark. Obnoxious code needed in system sort; not in this course (for brevity).

Duplicate keys

Often, purpose of sort is to bring items with equal keys together.

- Sort population by age.
- Find collinear points. ← see Assignment 3
- Remove duplicates from mailing list.
- Sort job applicants by college attended.

Typical characteristics of such applications.

- Huge array.
- Small number of key values.

```
Chicago 09:25:52
Chicago 09:03:13
Chicago 09:21:05
Chicago 09:19:46
Chicago 09:19:32
Chicago 09:00:00
Chicago 09:35:21
Chicago 09:00:59
Houston 09:01:10
Houston 09:00:13
Phoenix 09:37:44
Phoenix 09:00:03
Phoenix 09:14:25
Seattle 09:10:25
Seattle 09:36:14
Seattle 09:22:43
Seattle 09:10:11
Seattle 09:22:54
```

key

Duplicate keys

Mergesort with duplicate keys. Always between $\frac{1}{2} N \lg N$ and $N \lg N$ compares.

Quicksort with duplicate keys.

- Algorithm goes quadratic unless partitioning stops on equal keys!
- 1990s C user found this defect in `qsort()`.

several textbook and system implementation also have this defect

S T O P O N E Q U A L K E Y S

swap

if we don't stop on equal keys

if we stop on equal keys

Duplicate keys: the problem

Mistake. Put all items equal to the partitioning item on one side.

Consequence. $\sim \frac{1}{2} N^2$ compares when all keys equal.

B A A B A B B B C C C   **A A A A A A A A A A A**

Recommended. Stop scans on items equal to the partitioning item.

Consequence. $\sim N \lg N$ compares when all keys equal.

B A A B A B C C B C B   **A A A A A A A A A A A**

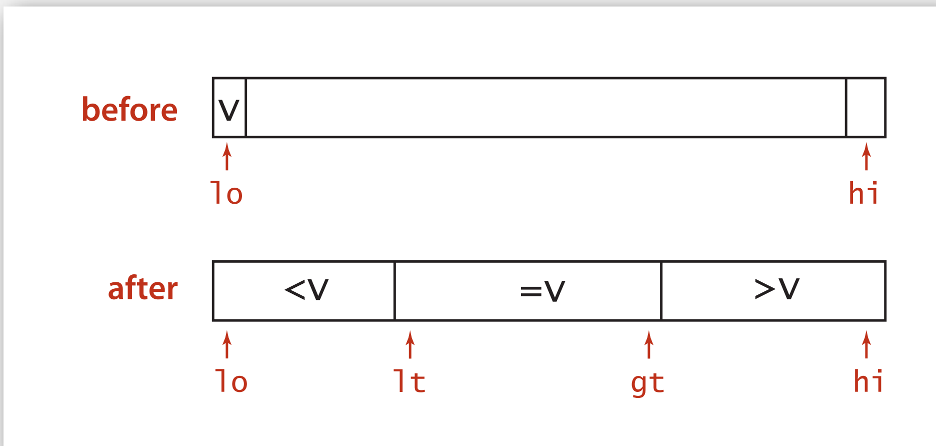
Desirable. Put all items equal to the partitioning item in place.

A A A B B B B B C C C   **A A A A A A A A A A A**

3-way partitioning

Goal. Partition array into 3 parts so that:

- Entries between `lt` and `gt` equal to partition item `v`.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.



Dutch national flag problem. [Edsger Dijkstra]

- Conventional wisdom until mid 1990s: not worth doing.
- New approach discovered when fixing mistake in C library `qsort()`.
- Now incorporated into `qsort()` and Java system sort.

Dijkstra's 3-way partitioning: demo

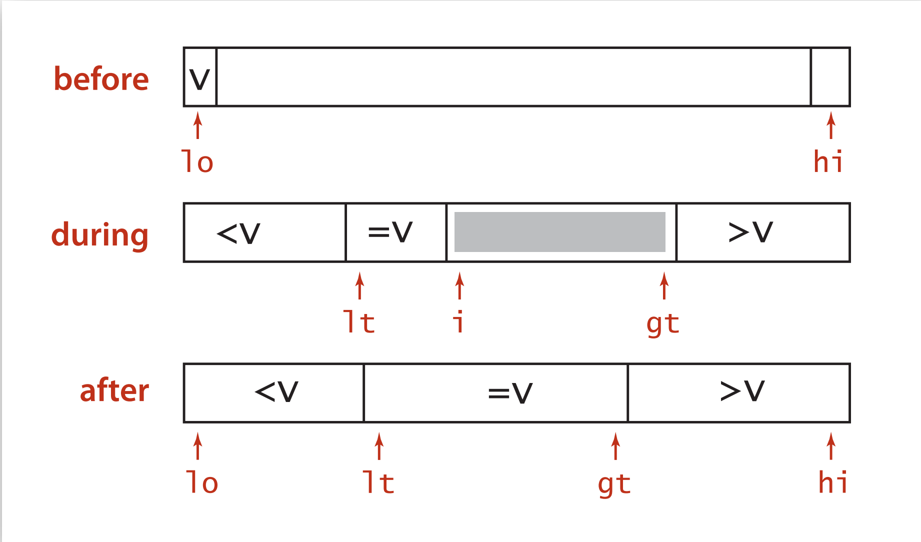
Dijkstra 3-way partitioning algorithm

3-way partitioning.

- Let `v` be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - `a[i]` less than `v`: exchange `a[lt]` with `a[i]` and increment both `lt` and `i`
  - `a[i]` greater than `v`: exchange `a[gt]` with `a[i]` and decrement `gt`
  - `a[i]` equal to `v`: increment `i`

Most of the right properties.

- In-place.
- Not much code.
- Linear time if keys are all equal.



Dijkstra's 3-way partitioning: trace

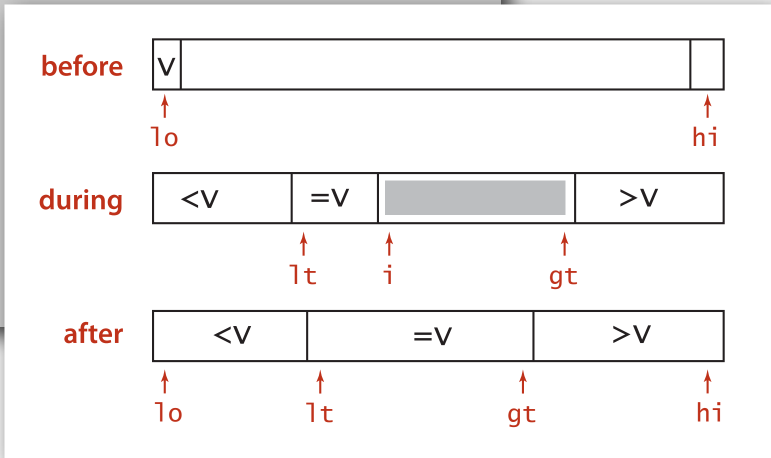
lt	i	gt	0	1	2	3	4	5	6	7	8	9	10	11
0	0	11	R	B	W	W	R	W	B	R	R	W	B	R
0	1	11	R	B	W	W	R	W	B	R	R	W	B	R
1	2	11	B	R	W	W	R	W	B	R	R	W	B	R
1	2	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	10	B	R	R	W	R	W	B	R	R	W	B	W
1	3	9	B	R	R	B	R	W	B	R	R	W	W	W
2	4	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	9	B	B	R	R	R	W	B	R	R	W	W	W
2	5	8	B	B	R	R	R	W	B	R	R	W	W	W
2	5	7	B	B	R	R	R	R	B	R	W	W	W	W
2	6	7	B	B	R	R	R	R	B	R	W	W	W	W
3	7	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W
3	8	7	B	B	B	R	R	R	R	R	W	W	W	W

3-way partitioning trace (array contents after each loop iteration)

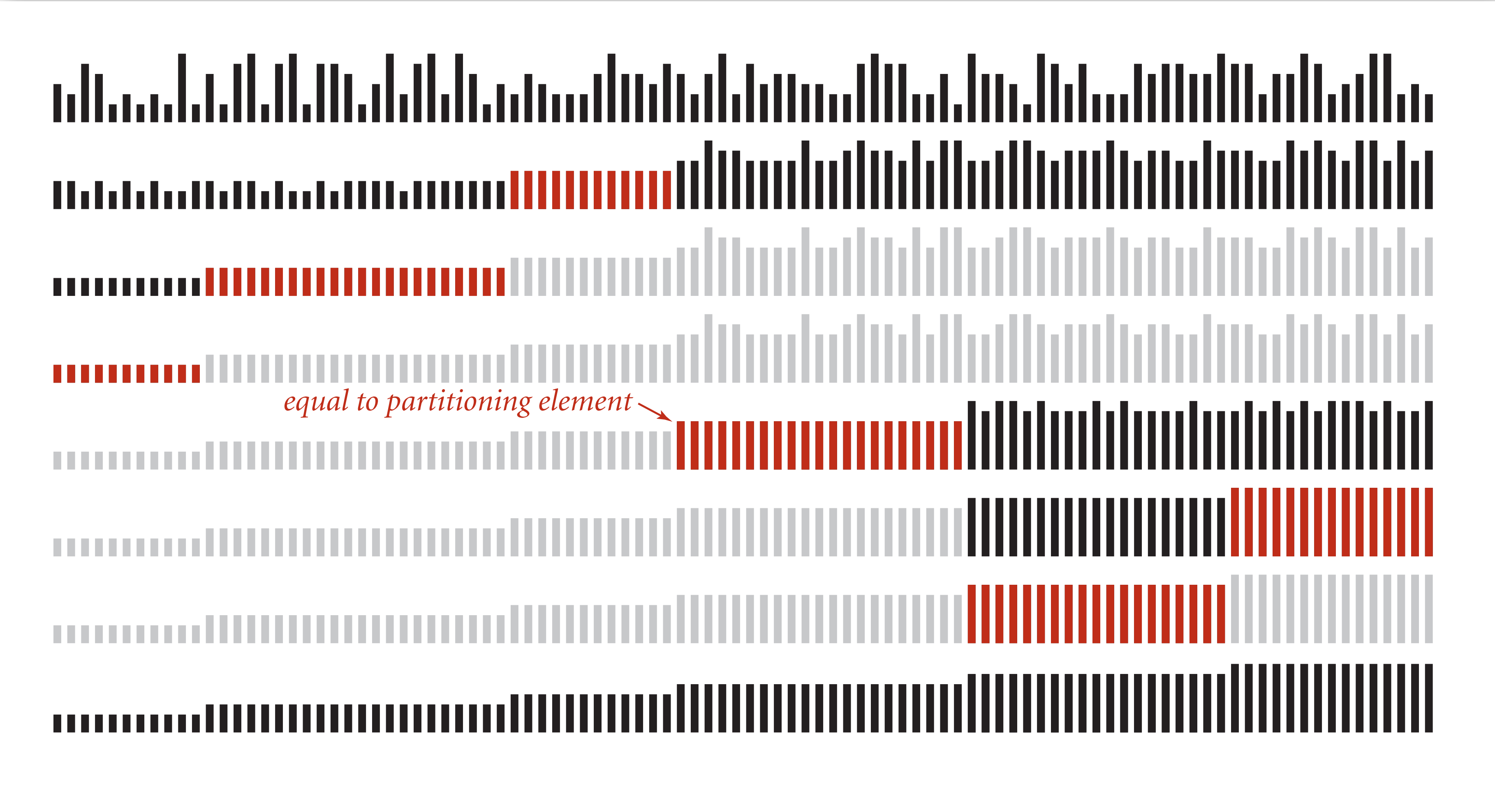
3-way quicksort: Java implementation

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   Comparable v = a[lo];
   int i = lo;
   while (i <= gt)
   {
      int cmp = a[i].compareTo(v);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



3-way quicksort: visual trace



Duplicate keys: lower bound

Sorting lower bound. If there are n distinct keys and the i^{th} one occurs x_i times, any compare-based sorting algorithm must use at least

$$\lg\left(\frac{N!}{x_1!\; x_2! \cdots x_n!}\right) \sim -\sum_{i=1}^{n} x_i \lg \frac{x_i}{N}$$

$N \lg N$ when all distinct;
linear when only a constant number of distinct keys

compares in the worst case.

Proposition. [Sedgewick-Bentley, 1997]
Quicksort with 3-way partitioning is entropy-optimal.

proportional to lower bound

Pf. [beyond scope of course]

Bottom line. Randomized quicksort with 3-way partitioning reduces running time from linearithmic to linear in broad class of applications.

- selection
- duplicate keys
- comparators
- system sorts

Sorting applications

Sorting algorithms are essential in a broad variety of applications:

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS feed in reverse chronological order.

obvious applications

- Find the median.
- Find the closest pair.
- Binary search in a database.
- Identify statistical outliers.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Supply chain management.
- Load balancing on a parallel computer.

. . .

non-obvious applications

Every system needs (and has) a system sort!

Java system sorts

`Arrays.sort()`.

- Has different method for each primitive type.
- Has a method for data types that implement `Comparable`.
- Has a method that uses a `Comparator`.
- Uses tuned quicksort for primitive types; tuned mergesort for objects.

```
import java.util.Arrays;

public class StringSort
{
   public static void main(String[] args)
   {
      String[] a = StdIn.readStrings());
      Arrays.sort(a);
      for (int i = 0; i < N; i++)
         StdOut.println(a[i]);
   }
}
```

Q. Why use different algorithms for primitive and reference types?

War story (C qsort function)

AT&T Bell Labs (1991). Allan Wilks and Rick Becker discovered that a `qsort()` call that should have taken a few minutes was consuming hours of CPU time.

At the time, almost all `qsort()` implementations based on those in:

- Version 7 Unix (1979): quadratic time to sort organ-pipe arrays.
- BSD Unix (1983): quadratic time to sort random arrays of 0s and 1s.

Engineering a system sort

Basic algorithm = quicksort.

- Cutoff to insertion sort for small subarrays.
- Partitioning scheme: Bentley-McIlroy 3-way partitioning. [ahead]
- Partitioning item.
  - small arrays: middle entry
  - medium arrays: median of 3
  - large arrays: Tukey's ninther [next slide]

Engineering a Sort Function

JON L. BENTLEY

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SUMMARY

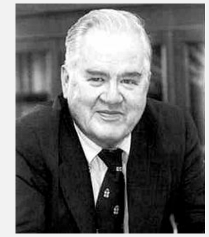
We recount the history of a new `qsort` function for a C library. Our function is clearer, faster and more robust than existing sorts. It chooses partitioning elements by a new sampling scheme; it partitions by a novel solution to Dijkstra's Dutch National Flag problem; and it swaps efficiently. Its behavior was assessed with timing and debugging testbeds, and with a program to certify performance. The design techniques apply in domains beyond sorting.

Now widely used. C, C++, Java, ….

Tukey's ninther

Tukey's ninther. Median of the median of 3 samples, each of 3 entries.

- Approximates the median of 9.
- Uses at most 12 compares.



nine evenly spaced entries	R	L	A	P	M	C	G	A	X	Z	K	R	B	R	J	J	E
groups of 3	R	A	M		G	X	K		B	J	E						
medians	M	K	E														
ninther	K																

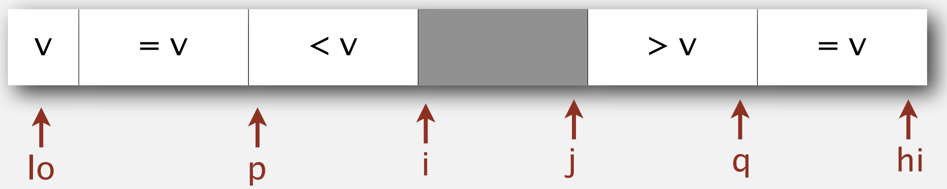
Q. Why use Tukey's ninther?

A. Better partitioning than random shuffle and less costly.

Bentley-McIlroy 3-way partitioning

Partition items into four parts:

- No larger entries to left of `i`.
- No smaller entries to right of `j`.
- Equal entries to left of `p`.
- Equal entries to right of `q`.



Afterwards, swap equal keys into center.

All the right properties.

- In-place.
- Not much code.
- Linear time if keys are all equal.
- Small overhead if no equal keys.

Achilles heel in Bentley-McIlroy implementation (Java system sort)

Q. Based on all this research, Java's system sort is solid, right?

more disastrous consequences in C

A. No: a killer input.

- Overflows function call stack in Java and crashes program.
- Would take quadratic time if it didn't crash first.

```
% more 250000.txt
0
218750
222662
11
166672
247070
83339
...
```

```
% java IntegerSort 250000 < 250000.txt
Exception in thread "main"
java.lang.StackOverflowError
   at java.util.Arrays.sort1(Arrays.java:562)
   at java.util.Arrays.sort1(Arrays.java:606)
   at java.util.Arrays.sort1(Arrays.java:608)
   at java.util.Arrays.sort1(Arrays.java:608)
   at java.util.Arrays.sort1(Arrays.java:608)
   ...
```

250,000 integers
between 0 and 250,000

Java's sorting library crashes, even if
you give it as much stack space as Windows allows

Achilles heel in Bentley-McIlroy implementation (Java system sort)

McIlroy's devious idea. [A Killer Adversary for Quicksort]

- Construct malicious input on the fly while running system quicksort, in response to the sequence of keys compared.
- Make partitioning item compare low against all items not seen during selection of partitioning item (but don't commit to their relative order).
- Not hard to identify partitioning item.

Consequences.

- Confirms theoretical possibility.
- Algorithmic complexity attack: you enter linear amount of data; server performs quadratic amount of work.

Good news. Attack is not effective if `sort()` shuffles input array.

Q. Why do you think `Arrays.sort()` is deterministic?

System sort: Which algorithm to use?

Many sorting algorithms to choose from:

Internal sorts.

- Insertion sort, selection sort, bubblesort, shaker sort.
- Quicksort, mergesort, heapsort, samplesort, shellsort.
- Solitaire sort, red-black sort, splaysort, Dobosiewicz sort, psort, ...

External sorts. Poly-phase mergesort, cascade-merge, oscillating sort.

String/radix sorts. Distribution, MSD, LSD, 3-way string quicksort.

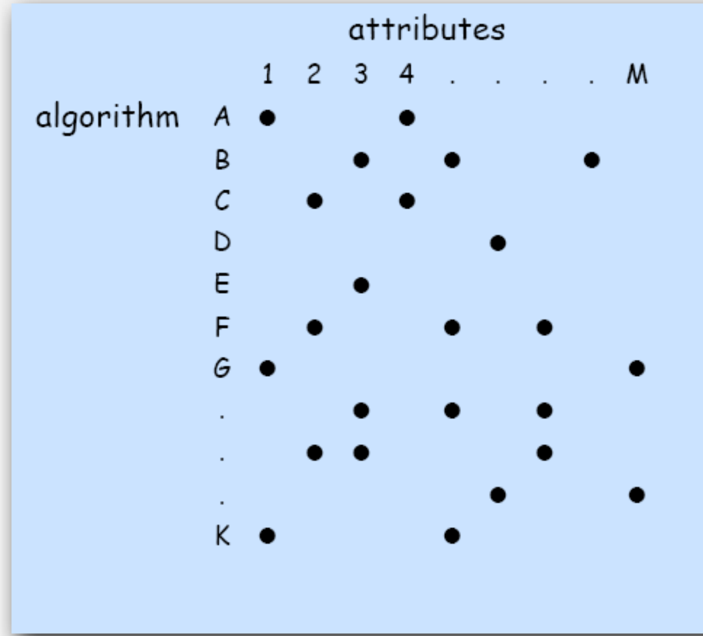
Parallel sorts.

- Bitonic sort, Batcher even-odd sort.
- Smooth sort, cube sort, column sort.
- GPUsort.

System sort: Which algorithm to use?

Applications have diverse attributes.

- Stable?
- Parallel?
- Deterministic?
- Keys all distinct?
- Multiple key types?
- Linked list or arrays?
- Large or small items?
- Is your array randomly ordered?
- Need guaranteed performance?



many more combinations of attributes than algorithms

Elementary sort may be method of choice for some combination.

Cannot cover all combinations of attributes.

Q. Is the system sort good enough?

A. Usually.

Sorting summary

	inplace?	stable?	worst	average	best	remarks
selection	✔		$N^2 / 2$	$N^2 / 2$	$N^2 / 2$	*N* exchanges
insertion	✔	✔	$N^2 / 2$	$N^2 / 4$	N	use for small *N* or partially ordered
shell	✔		?	?	N	tight code, subquadratic
merge		✔	N lg N	N lg N	N lg N	*N* log *N* guarantee, stable
quick	✔		$N^2 / 2$	2 N ln N	N lg N	*N* log *N* probabilistic guarantee fastest in practice
3-way quick	✔		$N^2 / 2$	2 N ln N	N	improves quicksort in presence of duplicate keys
???	✔	✔	N lg N	N lg N	N lg N	holy sorting grail